## Chapter - 2

### Linear Differential Equation

A linear differential equation of order $n$ is written as ~~a~~ $a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}}$

$$+ a_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots a_n(x)y = r(x)$$

where $a_0(x) \neq 0$, $a_0(x), a_1(x), \cdots a_n(x) r(x)$ can be function of $x$ only

$$\Rightarrow a_0(x)y^n(x) + a_1(x)y^{n-1}(x) + a_2(x)y^{n-2}(x)$$

$$+ \cdots + a_n(x)y = r(x).$$

# If $r(x) \neq 0$ then we call (1) as non-homogeneous linear differential equation.

If $a_0(x), a_1(x), a_2(x) \cdots a_n(x)$ are all constants coefficient then we say (1) is non-homogeneous ~~eq~~ linear differential eq. with constant coefficient otherwise we call it as non homogeneous linear Diff eq. with variable coefficient.

### $a_0(x)y^n(x) + a_1(x)y^{n-1}(x) + \dots + a_n(x)y = r(x)$

- $r(x) \neq 0$ → (NHLDE)
  (Non-Homogenous linear diff. eq.)
  - NHLDE with Constant Cofficient
  - NHLDE with variable Cofficient.
- $r(x) = 0$ → (HLDE)
  Homogenous linear Diff. eq.
  - HLDE with Constant Cofficient.
  - HLDE with Variable Cofficient.

eg. $y'' + 2y' + 3y = \sin x$ → $r(x)$.

$r(x) \neq 0$, So, it is NHLDE and differential Cofficient is 2 which is Constant. So, NHLDE with Constant Cofficient

# A linear differential equation of order $n$ is $a_0(x)y^n(x) + a_1(x)y^{n-1}(x) + a_2(x)y^{n-2}(x) \ldots a_{n-1}y'(x) + a_n(x)y = r(x)$

$r(x) \neq 0$

Non homogenious.

Theorem.

If the function $a_0(x), a_1(x), a_2(x) \ldots a_n(x), r(x)$ are continious over some interval $I$ and $a_0(x) \neq 0$ on $I$, then there exist a unique solution to the initial value problem.

$$a_0(y)^n(x) + a_1(y)^{n-1}(x) + a_2 y^{n-2}(x) + \ldots + a_n y(x) = r(x)$$

$$y(x_0) = c_1,\ y'(x_0) = c_2 \ldots y^{n-1}(x_0) = c_n$$

where $x_0 \in I$ and $c_1, c_2 \ldots c_n$ are all constraints.

# If above stated condition is satisfy then we say that diff. eq. is normal on $I$.

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# Any point $x_0$ where $a_0(x_0) \neq 0$ is called as ordinary or regular point.

# Any point $x_1$ where $a_0(x_1) = 0$ is called as linear point.

Ques: Find the interval on which following diff. eq. are normal?

(1) $y' = \frac{3y}{x}$ (ii) $(1+x^2)y'' + 2xy' + y = 0$

Sol^n. $x \frac{dy}{dx} - 3y = 0$

$a_0(x) = x, \quad a_1(x) = -3$

Here $a_0$ & $a_1$ are both continuous on [R]

and put $a_0((x)) = 0 \Rightarrow x = 0$

$\therefore$ Equation is normal sub. in the interval of $[-\infty, 0) \cup (0, \infty]$

(ii) $(1+x^2)y'' + 2xy' + y = 0$

there $a_0[x] = 1+x^2$, $a_1(x) = 2x$, $a_2(x) = 1$

$\therefore$ so $a_1, a_2$ are all continuous on R.

there $a_0(x) \neq 0$ on R.

$\therefore$ Given L.D.E is normal on any sub-interval. where of R or $[-\infty, \infty]$

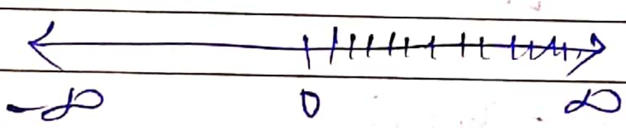
(iii) $y'' + 3y' + \sqrt{x}\,y = \sin x$.

$a_0(x) = 1$, $a_1(x) = 3$, $a_2(x) = \sqrt{x}$, $r(x) = \sin x$.

$a_0, a_1, \cancel{a_2}$ are continuous on $R$.

$a_2$ is continuous on $[0, \infty)$ and $r(x)$ is also continuous on $R$.

→ Common region of continuity.




Here $a_0, a_1, a_2, r(x)$ are all continuous on $[0, \infty)$ and $a_0 \neq 0$.

∴ Given ~~Heat~~ L.D.E is normal on any sub interval of $I = [0, \infty)$.

Ques: Find the interval on which following de are normal (i) $y''' + 9y' + y = \log(x^2 - 9)$

(ii) $y'' + |x|y' + y = x \log x$

(iii) $x(1-x)y'' - 3xy' - y = 0$

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Sol^n (i) $y''' + (0)y'' + 9y' + y = \log(x^2 - 9)$

$\rightarrow$ $a_0(x) = 1$, $a(x) = 0$, $a_2(x) = 9$, $a_3[x] = 1$

$r(x) = \log(x^2 - 9)$

Since $a_0, a, a_2, a_3$ are all constant, so they are continious over $R$

Now, $r(x) = \log[x^2 - 9] = \log[(x-3)(x+3)]$

$= \log(x+3) + \log(x-3)$ will be define if $x+3 > 0 \Rightarrow x > -3$

Here, It will continious on $[3, \infty)$ and $a_0(x) \neq 0$ on $[3, \infty)$.

Hence given LDE will be normal on any sub interval of the interval $I = [3, \infty)$

(ii) Home work

(iii) $x(x+1)y'' - 3y' - y = 0$

$a_0(x) = x(1-x)$, $a_1(x) = -3x$

$a_2(x) = -1$ and $r(x) = 0$

Here $a_0, a$ are polynomial function, so they are continious on $R$.

$a_2, r(x)$ are constant, so are continious on $R$.

Now,

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Now, $a_0 \neq 0$.

$x(1-x) = 0$
$x = 0, 1.$

$a_0(x) \neq 0$ for all $x \in \mathbb{R}$ other than $x = 0, 1.$

Hence, given LDE will be normal on any subinterval of the interval

$$I = (-\infty, 0) \cup (0, 1) \cup (1, \infty).$$

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* **Linear Combination of function**: Let $f_1(x), f_2(x) --- f_n(x)$ are $n$ function of $x$, then $c_1f_1 + c_2f_2 + --- + c_nf_n$ is called linear combination of $f_1, f_2 --- f_n$. where $c_1, c_2 --- c_n$ are constants.

\# Consider a homogeneous LDE with of order $n$, as $a_0(x)y^n(x) + a_1(x)y^{n-1}(x) + a_2(x)y^{n-2}(x) + --- + a_n(x)y(x) = 0$

Let $y_1, y_2 ---- y_n$ are $n$ solution of above LDE then their linear combination $c_1y_1 + c_2y_2 + --- + c_ny_n$ is also the solution of above LDE.

Ques: Verify given functions are solution of associated LDE and also verify that linear combination of these function is also a solution.

(1) $e^x, \; e^{-2x}$ ; $y'' + y' - 2y = 0$

Soln: Consider LDE $y'' + y' - 2y = 0 \Rightarrow \frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$ — (1)

Consider $y_1(x) = e^x$, put in (1).

$$\frac{d^2}{dx^2}[e^x] + \frac{d}{dx}(e^x) - 2(e^x)$$

$$e^x + e^x - 2e^x = 0 \Rightarrow 0 = 0$$

$\Rightarrow y_1(x) = e^x$ is the solution of (1)

again $y_2(x) = e^{-2x}$, put in (1)

$$\frac{d^2}{dx^2}[e^{-2x}] + \frac{d}{dx}(e^{-2x}) - 2e^{-2x} = 0$$

$$\Rightarrow e^{-2x}(-2)^2 - 2e^{-2x} - 2e^{-2x}$$

$$\Rightarrow \cancel{8}\, 4e^{-2x} - 2e^{-2x} - \cancel{2e}\, 2e^{-2x} = 0$$

$$\Rightarrow 0 = 0$$

$y_2 = e^{-2x}$ is also the solution of (1)

Now, Consider linear Combination of $y_1$ & $y_2$ as $y_3 = c_1 y_1 + c_2 y_2 = c_1 e^{x} + c_2 e^{-2x}$

Put $y_3$ in (1).

$\therefore$ (1) $\frac{d^2}{dx^2}\left[c_1 e^{x} + c_2 e^{-2x}\right] + \frac{d}{dx}\left[c_1 e^{x} + c_2 e^{-2x}\right]$
$-2\left[c_1 e^{x} + c_2 e^{-2x}\right] = 0$

$\Rightarrow c_1 e^{x} - 2c_2 e^{-2x} - 2c_1 e^{x} - 2c_2 e^{-2x} = 0$

$\Rightarrow c_1 e^{x} + 4c_2 e^{-2x} + c_1 e^{x} - 2c_2 e^{-2x} - 2c_1 e^{x} - 2c_2 e^{-2x} = 0$

$\Rightarrow 0 = 0.$

$\Rightarrow y_3 = c_1 y_1 + c_2 y_2$ is also the solution of LDE in (1).

H.W

Ques: $e^{-x}\cos 2x$, $e^{-x}\sin 2x$; $y'' + 2y' + 5y = 0$

follow the statement of previous ques. try to do in same

H.W

## Wronskian :

Let $f_1(x), f_2(x) --- f_n(x)$ are $n$ functions of $x$, then wronskian of $f_1, f_2, -- f_n$ is written as.

$$W(f_1, f_2, -- f_n) = \begin{vmatrix} f_1 & f_2 & -- & f_n \\ f_1' & f_2' & --- & f_n' \\ | & | & & | \\ f_1^{n-1} & f_2^{n-1} & -- & f_n^{n-1} \end{vmatrix}$$

# If $W(f_1, f_2 -- f_n) \neq 0$ for all $x \in I$ then $f_1, f_2, -- f_n$ are linearly Independent on $I$ and if $W(f_1, f_2 -- f_n) = 0$ for some $x_0 \in I$ then $f_1, f_2 -- f_n$ are linearly dependent.

# Let us consider a Homogeneous LDE of order $n$;

$a_0 y^n(x) + a_1 y^{n-1}(x) + a_2 y^{n-2}(x) -- + a_n y(x) = 0$

where $a_0, a_1 --- a_n$ are all function of $x$,

Let $y_1(x), y_2(x) -- y_n(x)$ are $n$ linear independent solution of above equation, then these solutions are called fundamental solution of the equation In that case, we can write general solution of above diff. eq. as $y(x) = c_1 y_1 + c_2 y_2 + -- c_n y_n$.

## These fundamental solutions are also called as basis of the diff. equation.

Ques: Examine whether following function are Independent for $x \in [0, \infty)$

(i) $2x, 6x+2, 3x+2$ (ii) $x^2-2x, 3x^2+x+2, 4x^2-x+1$
(iii) $\log x, \log x^2, \log x^3$

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(i) Here $f_1(x) = 2x$, $f_2(x) = 6x+2$,

$f_3(x) = 3x+2$.

Now, $w\{f_1, f_2, f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$

$= \begin{vmatrix} 2x & 6x+2 & 3x+2 \\ 2 & 6 & 3 \\ 0 & 0 & 0 \end{vmatrix}$

$= 0$

$\therefore f_1, f_2, f_3$ are linear independent on $(0,\infty)$.

(ii) $f_1 = x^2-2x$ $f_2 = 3x^2+x+2$ $f_3 = 4x^3-x+1$.

$w(f_1, f_2, f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f' & f_2' & f_3' \\ f'' & f_2'' & f_3'' \end{vmatrix}$

$\begin{vmatrix} x^2-2x & 3x^2+x+2 & 4x^2-x+1 \\ 2x-2 & 6x+1 & 8x-1 \\ 2 & 6 & 8 \end{vmatrix}$

$= 14$

$\Rightarrow w(f_1, f_2, f_3) = 14 \neq 0$ for $x \in (0,\infty)$

$= I$ $\therefore f_1,$

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$\therefore f_1, f_2, f_3$ are linearly independent on I.

# H.W Examine whether following function are linearly independent on $x \in [0, 2\pi]$.

(i) $x^2 - x,\ 3x^2 + x + 1,\ 9x^2 - x + 2$

(ii) $x - 1,\ x + 1,\ (x+1)^2$

(iii) $1,\ \cos x,\ \sin x$.

Ques: Show that in following problem given set of function for a fundamental solution to the adjoining diff. equation.

(i) $1, x^2$; $x^2 y'' - x y' = 0 \quad x > 0$ (1)

Sol^n Here $f_1 = 1 \qquad f_2 = x^2$

put $f_1$ into (1)

$$x^2(0) - x(0) = 0 \Rightarrow 0 = 0$$

$\Rightarrow f_1 = 1$ is the solution of (1)

Again, $f_2 = x^2$ put in (1)

$$(1)\ x^2[2] - x[2x] = 0 \Rightarrow 2x^2 - 2x^2 = 0$$
$$\Rightarrow 0 = 0.$$

$f_2$ is also solution of (1).

Now, $W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} 1 & x^2 \\ 0 & 2x \end{vmatrix} = 2x$

Now, $W(f_1, f_2) = 2x \neq 0$ if $x > 0$

$f_1, f_2$ are linearly independent.

$\Rightarrow f_1, f_2$ are fundamental solution if $x > 0$.

Ques: Show that the function $x, x^2, x^3$ are linearly independent any interval I, not containing zero.

Sol^n. $f_1 = x \quad f_2 = x^2, \quad f_3 = x^3$

Now, $W(f_1, f_2, f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$

$$= \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$= -2\left[3x^2 - x^3\right] + 6x\left[2x^2 - x^2\right]$$

$$= -2\left[2x^2\right] + 6x(x^2)$$
$$= 6x^3 - 4x^3 = 2x^3$$

$2x^3 = 0$

$W(f_1, f_2, f_3)$ will be zero on any I containing zero $(x = 0)$.

$\Rightarrow f_1, f_2, f_3$ will be linearly independent on any interval I. not containing zero.

**Ques:** Show that $x^{1/4}$, $x^{5/4}$ are the fundamental sets of solution for $16x^2y'' - 8xy' + 5y' = 0$

**Sol^n.** Let $y_1 = x^{1/4}$, put in (1)

$$\therefore \; 16x^2 \frac{d^2}{dx^2}(x^{1/4}) - 8x\frac{d}{dx}x^{1/4} + 5(x^{1/4}) = 0$$

$$16x^2\left(\frac{1}{4} \times \left(\frac{-3}{4}\right) x^{-7/4}\right) - 8x\left(\frac{1}{4}x^{-3/4}\right) + 5^{1/4} = 0$$

$$\Rightarrow -3x^{1/4} - 2x^{1/4} + 5x^{1/4} = 0 \Rightarrow 0 = 0$$

$y_1 = x^{1/4}$ is the solution of (1)

Now, $y_2 = x^{5/4}$ is the solution of (1)

Now, $y_2 = x^{5/4}$ put in (1)

$$\Rightarrow 16x^2\frac{d^2}{dx^2}(x^{5/4}) - 8x\frac{d}{dx}(x^{5/4}) + 5x^{5/4} = 0$$

$$\Rightarrow 16x^2 \times \frac{5}{4} - \frac{1}{4}x^{-3/4} - 8x\frac{5}{4}(x^{1/4}) + 5x^{5/4} = 0$$

$$\Rightarrow 5x^{5/4} - 10x^{5/4} + 5x^{5/4} = 0 \Rightarrow 0 = 0$$

$y_2 = x^{5/4}$ is also the solution of (1)

Now, $w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$

$$\begin{vmatrix} x^{1/4} & x^{5/4} \\ \frac{1}{4}x^{-3/4} & \frac{5}{4}x^{1/4} \end{vmatrix} \Rightarrow \frac{5}{4}x^{1/2} - \frac{1}{4}x^{1/2}$$

$$\sqrt{x} \neq 0.$$

Hence $y_1, y_2$ are fundamental solution.

H.W **Ques:** $e^{2x}\cos 3x$, $e^{2x}\sin 3x$, $2y'' - 8y' + 26y = 0$

check if given function are fundamental solution.

**Ques:** $e^x, e^{2x}, e^{-3x}$; $y''' - 3y'' - 4y' - 2y = 0$

\# If $y_1, y_2 -- y_n$ are $n$ fundamental solution of homogeneous LDE of order $n$ as, $a_0 y^n(x) + a_1 y^{n-1}(x) + a_2 y^{n-2}(x) + --- + a_n y(x) = 0$, $a_0 \neq 0$.

then general solution of this diff. eq.
$y = a_1 y_1 + a_2 y_2 + --- a_n y_n$, where $c_1, c_2 --- c_n$ are arbitary constant.

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Ques: Show that given set of function $y_1, y_2$ form a basis (fundamental solution) & hence solve the general problem.

(1) $e^x, e^{4x}$; $y'' - 5y' + 4y = 0$, $y(0) = 0$; $y'(0) = 1$

Sol^n. Here $y_1 = e^x$ putting in (1).

$$\frac{d^2}{dx^2}(e^x) - 5\frac{d}{dx}(e^x) + 4(e^x) = 0$$

$$e^x - 5e^x + 4e^x = 0 \Rightarrow 0 = 0.$$

$\Rightarrow y_1 = e^x$ is the solution of (1).

Again, $y_2 = e^{4x}$ put in (1)

$$(1) \Rightarrow \frac{d^2}{dx^2}(e^{4x}) - 5\frac{d}{dx}(e^{4x}) + 4(e^{4x}) = 0$$

$$\Rightarrow 16e^{4x} - 5\times 4e^{4x} + 4e^{4x} = 0$$

$$\Rightarrow 0 = 0$$

$y_2 = e^{4x}$ is also solution of (1).

Now, $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{4x} \\ e^x & 4e^{4x} \end{vmatrix}$

$$= 4e^{5x} - e^{5x} = 3e^{5x} \neq 0.$$

$\Rightarrow$ $y_1, y_2$ are linearly independent

$\Rightarrow$ $y_1, y_2$ are fundamental solution or basis of (1)

Hence, general solution of diff. eq. (I) is

$y_0 = c_1 y_1 + c_2 y_2 = c_1 e^{x} + c_2 e^{4x}$ — (II)

Now, given $y(0) = 2 \Rightarrow$ where $x = 0, y = 2$.
put in (II).

$c_1 e^0 + c_2 e^{4(0)} \Rightarrow c_1 + c_2 = 2$ — (III).
$\rightarrow x = 0$

Again, $y'(0) = 1 \Rightarrow$ when $x = 0 \Rightarrow y' = \frac{dy}{dx} = 1$

Now, $y' = \frac{dy}{dx} = c_1 e^{x} + 4c_2 e^{4x}$ — (IV)

Apply, $y' = 1$ when $x = 0$ in (IV).

$1 = c_1 e^0 + 4c_2 e^{4(0)} \Rightarrow c_1 + 4c_2 = 1$ (V)

(III) − (IV). $\Rightarrow -3c_2 = 1 \Rightarrow \boxed{c_2 = -1/3}$

$c_1 = 2 - c_2 = 2 - \left(\frac{-1}{3}\right) = \frac{7}{3}$ $\boxed{c_1 = \frac{7}{3}}$

Now, solution to initial value problems
is $y = \frac{7}{3} e^{x} - \frac{1}{3} e^{4x}$.

HW. Check the following eq. is given function are basis or not and if they are following basis, solve initial value problem (IVP)

(I) $e^{2x}, e^{-2x}$; $y'' - 4y = 0$, $y(0) = 1$, $y'(0) = 4$

(II) $x^2, 1/x^2$, $x^2 y'' + xy' - 4y = 0$, $y(1) = 2$,
$y'(1) = 6$.

# Solution of homogeneous linear differential equation with Constant Coefficient:

⇒ Consider a HLDEWC of order n, as

$$a_0 y^n(x) + a_1 y^{n-1}(x) + a_2 y^{n-2}(x) + \cdots + a_n y(x) = 0$$

$a_0 \neq 0$, $a_0, a_1, a_2 \cdots a_n$ are Constants.

↓

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2}}{dx^{n-2}} + \cdots a_n y = 0$$

take, $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$, $\cdots \frac{d^n}{dx^n} = D^n$ —

$$\Rightarrow a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \cdots + a_n y = 0$$

$$\Rightarrow [a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n] y = 0$$

Consider,

Auxiliary eq. [Characteristic eq] as.

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots a_n = 0$$

find the roots of this auxiliary equation as $m_1, m_2, \cdots m_n$.

* Case-1 : If all the roots $m_1, m_2, m_3 \ldots m_n$ are distinct & real. ∴ general solution of given HLDEWC is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots c_n e^{m_n x}$$

where $c_1, c_2 \ldots c_n$ are arbitrary constants.

* Case-2 :

If two roots are equal, that is $m_1 = m_2 = m$, $m_3, m_4, \ldots m_n$ and all roots are equal.

∴ General solution is

$$y = (c_1 + c_2 x) e^{mx} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Sub-case : If there three roots are equal.

$m_1 = m_2 = m_3 = m$, $m_4, m_5, m_6 \ldots m_n$ where are roots are real.

general solution is $y = (c_1 + c_2 x + c_3 x^2) e^{mx} + c_4 e^{m_4 x} + c_5 e^{m_5 x} + \ldots c_n e^{m_n x}$

Sub-Case (ii) : If four roots are equal.

$m_1 = m_2 = m_3 = m_4 = m$; $m_5, m_6 - - m_n$ are all real.

$\therefore$ General solution is:

$$y = (C_1 + C_2 x + C_3 x^2 + C_4 x^3) e^{mx} + C_5 e^{m_5 x} + C_6 e^{m_6 x} + - - - + C_n e^{m_n x}.$$

This can be general further in same.

Case-3 : (If roots are complex), let the roots are $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, $m_3, m_4 - - m_n$ where $m_1, m_2$ are complex & $m_3, m_4 - - - m_n$ are all real & distinct.

general solution,

$$y = e^{\alpha x} \{ C_1 \cos \beta x + C_2 \sin \beta x \} + [ C_3 e^{m_3 x} + C_4 e^{m_3 x} + C_4 e^{m_3 x} - - - C_n e^{m_n x} ]$$

Sub Case (1) { If complex roots repeat }:

$m_1 = m_2 = \alpha + i\beta$

$m_3 = m_4 = \alpha - i\beta$

general solution.

$$y = e^{\alpha x} \{ (C_1 + C_2 x) \cos \beta x + (C_3 + C_4 x) \sin \beta x \}$$

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Sub-Case : (ii)

$m_1 = m_2 = m_3 = \alpha + i\beta$

$m_4 = m_5 = m_6 = \alpha - i\beta$

General solution is $y =$

$$y(x) = e^{\alpha x}\{(c_1 + c_2 x + c_3 x^2)\cos\beta x + (c_4 + c_5 x + c_6 x^2)\sin\beta x\}$$

Ques: solve following differential equation.

(i) $y'' - 4y = 0$ (ii) $y'' + y' - 2y = 0$

(iii) $y'' + 4y' + 5y = 0$ (iv) $y'' + 2y' + y = 0$

Sol. $\frac{d^2y}{dx^2} - 4y' = 0$

$\frac{d}{dx} = D, \frac{d^2}{dx^2} = D^2$

$\Rightarrow D^2y - 4y = 0 \Rightarrow (D^2 - 4)y = 0$

Auxiliary eq. is $m^2 = 4 \Rightarrow (m-2)(m+2) = 0$

$m = 2, -2$

$\therefore$ General solution is

$y = c_1 e^{2x} + c_2 e^{-2x}$

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(ii) $y'' + y' - 2y = 0 \Rightarrow \frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$

Take $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$

$\Rightarrow D^2y + Dy - 2y = 0 \Rightarrow (D^2 + D - 2)y = 0$

Auxiliary eq. is $m^2 + m - 2 = 0 \Rightarrow (m+2)(m-1) = 0$

$\Rightarrow m = -2, 1.$

$\therefore$ General solution is $y = C_1e^{-2x} + C_2e^{x}$

(iii) $y'' + 4y' + 5y = 0$

$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$

Take $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$

$\Rightarrow D^2y + 4Dy + 5y = 0 \Rightarrow (D^2 + 4D + 5)y = 0$

Auxiliary eq. is $m^2 + 4m + 5 = 0$

$m^2 + 4m + 5 = 0$

$m = \frac{-4 \pm 2i}{2} = -2+i, -2-i$

General solution is.

$y = e^{-2x}[C_1\cos x + C_2\sin x]$

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(iv) Sol$^n$: $y'' + 2y' + y = 0$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

$$\frac{d}{dx} = D, \quad \frac{d^2}{dx^2} = D^2$$

$\Rightarrow D^2y + 2Dy + y = 0 \Rightarrow (D^2 + 2D + 1)y = 0$

Auxiliary eq. is $m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0$

$\Rightarrow m = -1, -1$ are the roots.

$\therefore$ General sol$^n$ $y = (C_1 + C_2x)e^{-x}$.

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Ques: $y'' + 25y = 0$, $y(0) = 1$, $y(\pi) = -1$
Solve boundary value problems

Sol$^n$: $\frac{d^2y}{dx^2} + 25y = 0$

Take $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$

$\Rightarrow D^2y + 25y = 0 \Rightarrow (D^2 + 25)y = 0$

Auxiliary eq. is $m^2 + 25 = 0 \Rightarrow m^2 = -25$

$m = \pm\sqrt{-25} = 5i, -5i$

General solution, $y = e^{0(x)}[c_1 \cos 5x + c_2 \sin 5x]$

$y = c_1 \cos 5x + c_2 \sin 5x$.

Now, $y(0) = 1 \Rightarrow y = 1$ when $x = 0$ put in (1).

$1 = c_1 \cos(5x \times 0) + c_2 \sin(5 \times 0) \Rightarrow c_1 = 1.$

Now, $y(\pi) = -1 \Rightarrow y = -1$ when $x = -\pi$, put in (1)

Ques: Solve initial value problem $y'' - y' - 12y = 0$, $y(0) = 4$, $y'(0) = -5$

Solⁿ. $y'' - y' - 12y = 0 \Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0$

$\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$

$\Rightarrow D^2y - Dy - 12y = 0 \Rightarrow (D^2 - D - 12)y = 0$

Auxiliary eq: $m^2 - m - 12 = 0 \Rightarrow (m-4)(m+3) = 0$

$m = 4, -3$

∴ General solution to diff. eq. is

$$y = c_1 e^{4x} + c_2 e^{-3x} \quad \text{— (i)}$$

Now, $y(0) = 4 \Rightarrow y = 4$ when $x = 0$ apply to (i)

$$4 = c_1 e^0 + c_2 e^0 \Rightarrow c_1 + c_2 \quad \text{— (ii)}$$

Now, $y' = \frac{dy}{dx} = c_1 e^{4x} - 3c_2 e^{-3x}$ — (iii).

Now, $y'(0) = -5 \Rightarrow y' = -5$ when $x = 0$, apply to (iii)

$$-5 = 4c_1 - 3c_2 \quad \text{— (iv).}$$

Now, From (ii) $c_2 = 4 - c_1$ ; put in (iv).

(iv) $\Rightarrow 4c_1 - 3(4 - c_1) = -5$

$\Rightarrow 4c_1 - 12 + 3c_1 = -5$

$\boxed{c_1 = 1}$ $\boxed{c_2 = 3}$

∴ Solution to initial value problem (IVP)

$$y = e^{4x} + 3e^{-3x}.$$

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Solve. Solve IVP, $4y'' + 12y' + 9y = 0$

Sol$^n$: $4\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 9y = 0$

$\frac{d}{dx} = D \quad \frac{d^2}{dx^2} = D^2$

$4D^2y + 12Dy + 9y = 0 \Rightarrow (4D^2 + 12D + 9)y = 0$

A'E is $4m^2 + 12m + 9 = 0$

$(2m+3)^2 = 0 \quad m = \frac{-3}{2}, \frac{-3}{2}$

∴ General solution to diff. eq. is $y = (C_1 + C_2x)$

$y = (C_1 + C_2x)e^{(-3/2)x}$ — (1)

Now $y(0) = -1 \Rightarrow y = -1$ when $x = 0$

apply to (1)

$-1 = [C_1 + C_2(0)]e^{-3/2(0)} = \boxed{C_1 = -1}$

diff (1) w.r.t. $x \Rightarrow \frac{dy}{dx} = y' = (C_1 + C_2) - e^{-3/2x}\left(\frac{-3}{2}\right) + e^{-3/2x}C_2$

Now, $y'(0) = 2 \Rightarrow y' = 2$ when $x = 0$

$2 = C_1\left(\frac{-3}{2}\right) + C_2 \Rightarrow C_2 = 2 + \frac{3}{2}C_1$

$= 2 + \frac{3}{2} \times (-1) = 1$

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$$\boxed{C_2 = \frac{1}{2}}$$

Now, solution to IVP is $y = \left(-1 + \frac{1}{2}x\right)e^{-3/2x}$

$$y = \left(-1 + \frac{1}{2}x\right)e^{-3/2x}$$

Ques: Find a diff equation of the form $ay'' + by' + cy = 0$ for which following function are the solution.

(i) $e^{3x}, e^{-2x}$ (ii) $e^{x/4}$,

(iii) $1, e^{-2x}$. (iv) $e^{2x}, xe^{2x}$

Solⁿ (i) Here solution to diff eq. are $e^{3x}, e^{-2x}$.

∴ Roots of axiliary eq. are $3, -2$.
Auxiliary eq. $(m-3)(m+2) = 0$

$$m^2 - m - 6 = 0$$

operator is $(D^2 - D - 6)$

$\Rightarrow$ Diff. eq. $(D^2 - D - 6)y = 0$

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

$$y'' - y' - 6y = 0$$

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(iii) Solution of diff. eq. is $1, e^{-2x}$
Roots of auxiliary eq. are $0, -2$.

Auxiliary eq. $(m-0)(m+2) = 0$
$m^2 + 2m = 0$
Operator is $(D^2 + 2D)$

Diff. eq. $(D^2 + 2D)y = 0$

$$\frac{d^2y}{dx^2} + \frac{2dy}{dx} = 0 \Rightarrow y'' + 2y' = 0$$

(iv) Solution of diff. eq. are $e^{2x}, xe^{2x}$.
Roots of auxiliary eq. are $2, 2$.

Auxiliary eq. $(m-2)(m-2) = 0$
$m^2 - 4m + 4 = 0$
operator is $(D^2 - 4D + 4)$

Diff. eq. is $(D^2 - 4D + 4)y = 0$

$$\frac{d^2y}{dx^2} - \frac{4dy}{dx} + 4y = 0 \Rightarrow y'' - 4y' + 4y = 0$$

(v) Solution to Diff. eq. are $e^{(5+3i)x}, e^{(5-3i)x}$

$\therefore$ Roots of A.E are $(5+3i)(5-3i)$
$(m-5-3i)(m-5+3i) = 0$
$m^2 + 25 - 10m + 9 = 0$
$m^2 - 10m + 34 = 0$

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operator is $(D^2 - 10D + 34)$

Diff. eq. is $(D^2 - 10D + 34)y = 0$

$$y'' - 10y' + 34y = 0.$$

**Ques:** Find general solution of following diff. eq.

(i) $2y''' + y'' - 13y' + 6y = 0$

(ii) $y''' + 4y'' + 5y' + 2y = 0$

**Sol.** $2y''' + y'' - 13y' + 6y = 0$

$$2\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - 13\frac{dy}{dx} + 6y = 0.$$

$$D = \frac{d}{dx},\ D^2 = \frac{d^2}{dx^2},\ D^3 = \frac{d^3}{dx^3}$$

$\Rightarrow$ $2D^2y + D^2y - 13Dy + 6y = 0$

$$[2D^3 + D^2 - 13D + 6]y = 0$$

Auxiliary eq. is $2m^3 + m^2 - 13m + 6 = 0$

$$m = 2, -3, 1/2$$

$\therefore$ General solution to given diff. eq.

$$y = C_1 e^{2x} + C_2 e^{-3x} + C_3 e^{\frac{1}{2}x}$$

Ques: $y''' + 4y'' + 5y' + 2y = 0$

$$D^3y + 4D^2y + 5Dy + 2y = 0$$

A.E $\rightarrow$ $m^3 + 4m^2 + 5m + 2 = 0$.

$$m = -1, -1, -2.$$

$\therefore$ General solution is

$$y = (C_1 + C_2 x)e^{-x} + C_3 e^{-2x}.$$

Ques: solve $y^{iv} - 13y'' + 36y = 0$

Sol $\rightarrow$ $$\frac{d^4y}{dx^4} - 13\frac{d^2y}{dx^2} + 36y = 0$$

$$\frac{d}{dx} = D, \quad \frac{d^2}{dx^2} = D^2$$

$$D^4y - 13D^2y + 36y = 0$$

$$[D^4 - 13D^2 + 36]y = 0$$

A.E. is $m^4 - 13m^2 + 36 = 0$

$m^2 = t$

$\therefore t^2 - 13t + 36 = 0$

$t = 9, 4, \quad m = 3, -3, 2, -2$

G.S is $y = C_1 e^{2x} + C_2 e^{-2x} + C_3 e^{3x} + C_4 e^{-3x}$

**Ques:** Solve $4y^{iv} + 4y''' - 3y'' - 2y' + y = 0$

**Sol.** $4\frac{d^4y}{dx^4} + 4\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

$= 4D^4y + 4D^3y - 3D^2y - 2Dy + y = 0$

$[4D^4 + 4D^3 - 3D^2 - 2D + 1]y = 0$

**A.E:** $4m^4 + 4m^3 - 3m^2 - 2m + 1 = 0$

Put $m = -1$

$4 - 4 - 3 + 2 + 1 = 0 = 0$

$m = -1, -1, 1/2, 1/2$

Solution is $y = (C_1 + C_2 x)e^{-x} + (C_3 x + C_4 x) e^{x/2}$

**Ques:** $y''' + 4y'' + 5y' + 2y = 0$

**Sol$^n$** $D^3y + 4D^2y + 5Dy + 2y = 0$

A·E, $m^3 + 4m^2 + 5m + 2 = 0$

~~Take m~~

$m = -1, -1, -2.$

Solution is.

$y = c_1 e^{-x} + c_2 x e^{-x}, c_3 e^{-2x}$

Ques: $y''' - 2y'' + 4y' - 8y = 0$

Sol^n. A·E $= m^3 - 2m^2 + 4m - 8 = 0$

$m = 2, 2i, -2i$

G·S is $y = c_1 e^{2x} + e^{(0)x}[c_2 \cos(2x) + c_3 \sin 2x]$

$y = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x.$

Ques: $y^{iv} + 8y'' - 9y = 0$

Sol^n A-E is $m^4 + 8m^2 - 9 = 0$

$m = 3i, -3i, 1, -1.$

G-S = $y = c_1 e^{x} + c_2 e^{-x} + c_3 \cos 3x + c_4 \sin 3x.$

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Ques: find a HLDE with real coefficient of lowest degree which has following particular solution.

(i) $5 + e^{x} + 3e^{3x}$.

Roots of A.E for that HLDEWC are $m = 3, 1, 0$

$$A.E = (m-3)(m-1)(m-0) = 0$$
$$m^3 - 4m^2 + 3m = 0$$

$$D.E. = y''' - 4y'' + 3y' = 0.$$

(ii) $e^{-x} + \cos 5x + 3\sin 5x$,

$$AE = m = -1, 5i, -5i$$
$$(m+1)(m-5i)(m+5i) = 0$$
$$m^3 + m^2 + 25m + 25 = 0$$

HLDEWC is $y''' + y'' + 25y' + 25y = 0$

(iii) Particular solution to HLDEWC is $xe^{-x} + e^{2x}$

Roots of AE are $m = 2, -1, -1$

$$AE \text{ is } (m-2)(m+1)(m+1) = 0$$
$$m^3 - 3m - 2 = 0$$

$\therefore$ HLDEWC is $y''' - 3y' - 2y = 0$

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(iv) $1+x+e^{x}-3e^{3x}$.

Roots of are A.E are $3, 1, 0, 0 = m$.

$$(m-3)(m-1)(m-0)(m-0)=0$$

$$m^4-4m^3+3m^2=0$$

HLDEWC is $y^{iv}-4y'''+3y''=0$

(v) $x^2e^{2x}+2e^{-2x}$.

Roots of A.E are $-2, 2, 2, 2$.

AE is $(m+2)\{m^3-8-6m^2+12m\}$.

$= m^4-4m^3+16m-16=0$

$D.E = y^{iv}-4y'''+16y'-16y=0$

(vi) $3\cos 2x+5\sinh[3x]$

$3\cos 2x+5\left[\frac{e^{3x}-e^{-3x}}{2}\right]$

$3\cos 2x+\frac{5}{2}e^{3x}-\frac{5}{2}e^{-3x}$.

$\sinh\theta=\frac{e^{\theta}-e^{-\theta}}{2}$

$\cosh\theta=\frac{e^{\theta}+e^{-\theta}}{2}$

$AE = m = 3, -3, 2i, -2i$

$(m-3)(m+3)(m-2i)(m+2i)$

$= m^4-5m^2-36=0$

D.E is $y^{iv}-5y''-36y=0$.

Ques: Solve IVP $y''' - 2y'' - 5y' + 6y = 0$; $y(0) = 0$

$y'(0) = 0$ $y''(0) = 1$

Sol$^n$ $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

A·E $m^3 - 2m^2 - 5m + 6 = 0$

∴ Take $m = 1$

$m = +1, 3, -2$

$(m-1)(m-3)(m+2)$

G·S· is $y = c_1 e^x + c_2 e^{-2x} + c_3 e^{3x}$

$y(0) = c_1 + c_2 + c_3 = 0$

$y'(0) = c_1 e^x - 2c_2 e^{-2x} + 3c_3 e^{3x}$

$y(0) = c_1 - 2c_2 + 3c_3 = 0$

$y''(0) = c_1 + 4c_2 + 9c_3 = 1$

$0 + 3c_2 - 2c_3 = 0$

$-6c_2 - 6c_3 = 1$

~~$-3c_2$~~

$$\cancel{6C_2} - 4C_3 = 0$$
$$-6C_2 - 6C_3 = -1$$

$$-10\, C_3 = -1$$

$$\boxed{C_3 = \frac{1}{10}}$$

(1) $$3C_2 - 2 \times \frac{1}{10} = 0$$

$$3C_2 = \frac{1}{5}$$

$$\boxed{C_2 = \frac{1}{15}}$$

$$C_1 = 2 \times \frac{1}{15} - 3 \times \frac{1}{10}$$

$$= \frac{2}{15} - \frac{3}{10} = \frac{4-9}{30} = -\frac{5}{30}$$

$$\boxed{C_1 = \frac{-1}{6}}$$

$$-\frac{1}{6} + \frac{1}{10} + \frac{1}{15} = \frac{-5\ +1+}{30}$$

solution of given P, $y = -\frac{1}{6} e^{x} + \frac{1}{15} e^{-2x} + \frac{1}{10} e^{3x}$

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**Ques:** Solve BVP, $y''' + \pi^2 y' = 0$ , $y(0), y(1) = 0$

$$y'(0) + \cancel{b}y'(1) = 0.$$

**Sol:**

$$\frac{d^3y}{dx^3} + \pi^2 \frac{dy}{dx} = 0$$

$$AE = m^3 + \pi^2 m = 0$$

$$m = 0, \pm \pi i$$

G.S $\quad y = C_1 e^{(0)x} + e^{(0)x}\{C_2 \cos \pi x + C_3 \sin \pi x\}$

$$y = C_1 + C_2 \cos \pi x + C_3 \sin \pi x \text{ --- (I)}$$

$$y' = -\pi C_2 \sin \pi x + \pi C_3 \cos \pi x \text{ --- (II)}$$

$$\boxed{C_1 = 0} \quad \boxed{C_2 = 0}$$

$$y' = \pi C_3 \cos \pi x \quad [\because C_2 = 0]$$

Now, $y'(0) + y'(1) = 0$

$$\pi C_3 \cos 0 + \pi C_3 \cos \pi = 0 \Rightarrow \pi C_3 - C_3 \pi = 0$$

Solution of BVP is $y = C_3 \sin \pi x$,

$C_3$ can take any real value.